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New families of flows between two-dimensional conformal field theories

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Abstract

We present evidence for the existence of infinitely-many new families of renormalisation group flows between the nonunitary minimal models of conformal field theory. These are associated with perturbations by the ϕ_{21} and ϕ_{15} operators, and generalise a family of flows discovered by Martins. In all of the new flows, the finite-volume effective central charge is a non-monotonic function of the system size. The evolution of this effective central charge is studied by means of a nonlinear integral equation, a massless variant of an equation recently found to describe certain massive perturbations of these same models. We also observe that a similar non-monotonicity arises in the more familiar ϕ_{13} perturbations, when the flows induced are between nonunitary minimal models.

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1 Introduction

Integrable quantum field theories which interpolate between different conformal field theories have been recognised as an intriguing feature of two-dimensional models ever since the initial paper of A.B. Zamolodchikov [1]. The first examples arose as ϕ_{13} perturbations of unitary minimal models $\mathcal{M}_{p,p+1}$. The existence of higher-spin integrals of motion [2] indicates that such perturbations should be integrable; in [1] (see also [3]) perturbative arguments were used to show that for one sign of the coupling constant the resulting flow is to the neighbouring unitary minimal model, namely $\mathcal{M}_{p-1,p}$, at least for $p \gg 1$. Further support for this picture, this time valid at all values of p , was provided when ideas from the thermodynamic Bethe ansatz (TBA) enabled exact integral equations describing the flows to be proposed [4].

Unitarity of a conformal field theory is not a necessary requirement for the existence of integrable perturbations, and in [5, 6] the arguments of [1] were extended to the ϕ_{13} perturbations of more general minimal models $\mathcal{M}_{p,q}$. The results indicated that there should exist integrable flows between the models $\mathcal{M}_{p,q}$ and $\mathcal{M}_{2p-q,p}$, at least in the region $p \gg q-p$ where calculations perturbative in $(q-p)/p$ can be trusted. (It remains an open problem to give a TBA treatment of the nonunitary flows, which would not suffer from this caveat.) The unitary and nonunitary models can be put on a common footing by defining a parameter $\zeta = p/(q-p)^*$. The value of ζ suffices to identify p and q uniquely, since they must be coprime. In all cases the predicted flow is from the model specified by ζ to the model specified by $\zeta-1$.

Many further massless flows have since been discovered and studied, often by means of the TBA technique: the papers [7–19] are a sample of this work. Our main interest here will be the flows found by Martins [12] and further studied by Ravanini et al. [19]. These are perturbations of the minimal models $\mathcal{M}_{p,2p-1}$ and $\mathcal{M}_{p,2p+1}$ by the operators ϕ_{21} and ϕ_{15} respectively, and the predicted trajectories are:

$$\phi_{21} : \mathcal{M}_{p,2p-1} \rightarrow \mathcal{M}_{p-1,2p-1} ; \quad (1.1)$$

$$\phi_{15} : \mathcal{M}_{p,2p+1} \rightarrow \mathcal{M}_{p,2p-1} . \quad (1.2)$$

These flows can be chained together to form a single sequence, along which the perturbing operator alternates between ϕ_{21} and ϕ_{15} .

In this paper we will extend this picture by proposing further sequences of flows for which the perturbing operator alternates in the same way. As in studies based on the TBA technique, our main tool will be a conjecture for an exact equation expressing the finite-volume ground state energy of each model as a function of the system size R . In fact, it will be most convenient to work with an associated scaling function known as the ‘effective central charge’, $c_{\text{eff}}(r)$. This is related to the ground state energy as

$$E_0(M, R) = E_{\text{bulk}}(M, R) - \frac{\pi}{6R} c_{\text{eff}}(r) , \quad r = MR . \quad (1.3)$$

* ζ was denoted m in [5]; the symbol p is also often used, but we are reserving this for the integer-valued first index of the minimal models $\mathcal{M}_{p,q}$

In this equation, E_0 is the ground state energy, E_{bulk} the irregular bulk part, and M some mass scale, set for example by an infinite-volume one-particle state. The effective central charge depends on the system size only through the dimensionless combination MR ; for a model which happens to be scale invariant (conformal), it is constant and equal to $c-24d$, where c is the model's central charge and d its lowest conformal dimension. The previously-studied sequence (1.1), (1.2) is picked out by the monotonicity of the function c_{eff} as a function of r . In all other cases, somewhat to our surprise, we found that the effective central charge undergoes a number of oscillations as it interpolates between its short and long distance limits.

Our approach differs from the TBA method, and is much closer to that of the papers [20–26], in that a single nonlinear integral equation (NLIE) is proposed to describe infinitely-many different perturbed conformal field theories, each being picked out by an appropriate choice of certain parameters. For massless flows, the one previous example of such an equation was found by Al. Zamolodchikov [20], and his paper formed a large part of the motivation for our work.

The particular nonlinear integral equation that we will be using is obtained in §2; then in §3 we discuss the nature of the flows that it predicts, and in §4 we take a more detailed look at various asymptotics. This allows us to back up our conjectures with a comparison with UV conformal perturbation theory. In §5 we comment on some features of the ϕ_{13} flows, and in the concluding §6 we indicate some open problems that remain for future work.

2 The nonlinear integral equation

The work of [20] built in part on the nonlinear integral equation of [21][†], which encodes the finite-volume ground state energy of the sine-Gordon model for a continuous range of the coupling β , and at general ‘twist’ α . Setting $\zeta = \beta^2/(8\pi - \beta^2)$, it can be shown that for $\zeta = p/(q-p)$ and $\alpha = 1/p$, the ground-state energy of the ϕ_{13} perturbation of a general minimal model $\mathcal{M}_{p,q}$ is also matched [23, 24, 27–29]. (Note that this definition of ζ is thus in line with the one given in the introduction.) Since the sine-Gordon model is massive, the minimal model perturbations reproduced in this way are also massive. In contrast, the modification to the equation found in [20] describes a massless perturbation of the sine-Gordon model, flowing from the model at ζ to the model at $\zeta-1$, and thus matching the pattern of massless ϕ_{13} flows discussed in the introduction. While only the $\alpha = 0$ case was treated explicitly in [20], we will confirm below that with a suitable nonzero value of α the massless ϕ_{13} perturbations of minimal models are also obtained, with results that agree with the perturbative picture. In the nonunitary cases this yields some genuinely new information, since, as already mentioned, TBA systems giving the exact evolution of the ground-state energy for these models are not known.

However this is not the main purpose of our paper: rather, we would like to perform a

[†]the scale-invariant form of this equation had arisen previously in the context of integrable lattice models [22]

similar trick for the ϕ_{21} and ϕ_{15} perturbations of minimal models. Recently, a nonlinear integral equation describing finite size effects in the $a_2^{(2)}$ model was conjectured [25][‡], and checked to describe the massive ϕ_{12} , ϕ_{21} and ϕ_{15} perturbations of minimal models on the imposition of a suitably-chosen twist. (The connection between the $a_2^{(2)}$ model and these perturbations can be understood through quantum group reduction [30–32].) Our aim is to modify this equation so as to find out about the massless, interpolating, flows that may also be induced by these perturbations.

The nonlinear integral equation found in [25] is

$$f(\theta) = i\pi\alpha - ir \sinh \theta + \int_{\mathcal{C}_1} \varphi(\theta - \theta') \ln(1 + e^{f(\theta')}) d\theta' - \int_{\mathcal{C}_2} \varphi(\theta - \theta') \ln(1 + e^{-f(\theta')}) d\theta' \quad (2.1)$$

with the effective central charge given in terms of its solution as

$$c_{\text{eff}}(r) = \frac{3ir}{\pi^2} \left(\int_{\mathcal{C}_1} \sinh \theta \ln(1 + e^{f(\theta)}) d\theta - \int_{\mathcal{C}_2} \sinh \theta \ln(1 + e^{-f(\theta)}) d\theta \right). \quad (2.2)$$

The contours \mathcal{C}_1 and \mathcal{C}_2 run from $-\infty$ to $+\infty$, just below and just above the real θ -axis, and r is equal to MR , R being the size of the system and M the mass scale, set by the fundamental kink. The kernel

$$\varphi(\theta) = - \int_{-\infty}^{\infty} \frac{e^{ik\theta} \sinh(\frac{\pi}{3}k) \cosh(\frac{\pi}{6}k(1-2\xi)) dk}{\cosh(\frac{\pi}{2}k) \sinh(k\frac{\pi}{3}\xi)} \frac{dk}{2\pi} \quad (2.3)$$

is equal to $i/2\pi$ times the logarithmic derivative of the scalar factor in the S-matrix of the $a_2^{(2)}$ (Tzitzéica-Izergin-Korepin-Bullough-Dodd-Zhiber-Mikhailov-Shabat ...) model. The parameter ξ , related to the $a_2^{(2)}$ coupling γ as $\xi = \gamma/(2\pi - \gamma)$, is the analogue of the sine-Gordon parameter ζ mentioned earlier, while α corresponds to the twist. To obtain massive ϕ_{12} , ϕ_{21} and ϕ_{15} perturbations of a minimal model $\mathcal{M}_{p,q}$, the values of ξ and α must be chosen as follows [25]:

$$\phi_{12} \quad : \quad \xi = \frac{1}{\frac{2q}{p} - 1} \quad , \quad \alpha = 2/p \quad , \quad p < 2q ; \quad (2.4)$$

$$\phi_{21} \quad : \quad \xi = \frac{1}{\frac{2p}{q} - 1} \quad , \quad \alpha = 2/q \quad , \quad p > q/2 ; \quad (2.5)$$

$$\phi_{15} \quad : \quad \xi = \frac{1}{\frac{q}{2p} - 1} \quad , \quad \alpha = 1/p \quad , \quad p < q/2 ; \quad (2.6)$$

In each case, the inequality delimits the region of the (p, q) plane within which ξ is positive, and the corresponding perturbation is relevant. (The first inequality is not strictly necessary, since we will be adopting the convention that $p < q$ in the specification

[‡]as for the sine-Gordon model, a scale invariant version of this equation had previously been found in the context of integrable lattice models [26]

of $\mathcal{M}_{p,q}$.) The UV effective central charge, $c_{\text{eff}}(0) = 1 - \frac{3\xi}{\xi+1}\alpha^2$, is always equal to $1 - 6/pq$, but the subsequent terms in the expansions differ, as expected given the different perturbations being described.

The modification to the massive equation of [21] found in [20] made use of elements of the corresponding massless scattering theory, proposed in [33]. In our case we do not have a description of the massless scattering in terms of an S-matrix but, proceeding by analogy with the results of [20], we will substitute the single equation (2.1) with two equations describing hypothetical left and right movers. From the formulae (2.5) and (2.6), we notice that the massive versions of the perturbations (1.1) and (1.2) have $\xi = 2p - 1$ and $\xi = 2p$ respectively. We therefore seek a massless modification of the NLIE (2.1) which will interpolate in general between an ultraviolet theory with parameter ξ and an infrared theory with parameter $\xi - 1$. To this end, we introduce two analytic functions $f_R(\theta)$ and $f_L(\theta)$, couple them together via

$$\begin{aligned} f_R(\theta) = & -i\frac{r}{2}e^\theta + i\pi\alpha' \\ & + \int_{\mathcal{C}_1} \phi(\theta - \theta') \ln(1 + e^{f_R(\theta')}) d\theta' - \int_{\mathcal{C}_2} \phi(\theta - \theta') \ln(1 + e^{-f_R(\theta')}) d\theta' \\ & + \int_{\mathcal{C}_1} \chi(\theta - \theta') \ln(1 + e^{-f_L(\theta')}) d\theta' - \int_{\mathcal{C}_2} \chi(\theta - \theta') \ln(1 + e^{f_L(\theta')}) d\theta' \quad (2.7) \end{aligned}$$

$$\begin{aligned} f_L(\theta) = & -i\frac{r}{2}e^{-\theta} - i\pi\alpha' \\ & + \int_{\mathcal{C}_2} \phi(\theta - \theta') \ln(1 + e^{f_L(\theta')}) d\theta' - \int_{\mathcal{C}_1} \phi(\theta - \theta') \ln(1 + e^{-f_L(\theta')}) d\theta' \\ & + \int_{\mathcal{C}_2} \chi(\theta - \theta') \ln(1 + e^{-f_R(\theta')}) d\theta' - \int_{\mathcal{C}_1} \chi(\theta - \theta') \ln(1 + e^{f_R(\theta')}) d\theta' \quad (2.8) \end{aligned}$$

and replace the expression (2.2) for the effective central charge with

$$\begin{aligned} c_{\text{eff}}(r) = & \frac{3ir}{2\pi^2} \left[\int_{\mathcal{C}_1} e^\theta \ln(1 + e^{f_R(\theta)}) d\theta - \int_{\mathcal{C}_2} e^\theta \ln(1 + e^{-f_R(\theta)}) d\theta \right. \\ & \left. + \int_{\mathcal{C}_2} e^{-\theta} \ln(1 + e^{f_L(\theta)}) d\theta - \int_{\mathcal{C}_1} e^{-\theta} \ln(1 + e^{-f_L(\theta)}) d\theta \right]. \quad (2.9) \end{aligned}$$

As before, the contours \mathcal{C}_1 and \mathcal{C}_2 run from $-\infty$ to $+\infty$ just above and just below the real axis, and $r = MR$. However, since the flows are massless M no longer has a direct interpretation as the mass of an asymptotic state, but rather sets the crossover scale.

In the far infrared, $r \rightarrow \infty$ and the two equations (2.7) and (2.8) decouple. The result is two copies of the UV limit of (2.1), with the kernel $\varphi(\theta)$ substituted by $\phi(\theta)$. Since the IR destination is to be the model at $\xi - 1$, we take $\phi(\theta)$ to be the massive kernel (2.3) with the substitution $\xi \rightarrow \xi - 1$:

$$\phi(\theta) = - \int_{-\infty}^{\infty} \frac{e^{ik\theta} \sinh(k\frac{\pi}{3}) \cosh(\frac{\pi}{6}k(3-2\xi)) dk}{\cosh(\frac{\pi}{2}k) \sinh(k\frac{\pi}{3}(\xi-1))} \frac{dk}{2\pi}. \quad (2.10)$$

To find the other kernel, $\chi(\theta)$, we consider the UV behaviour of the new system. This should coincide with the UV limit of the massive system (2.1). As $r \rightarrow 0$, the solution to (2.7), (2.8) splits in the standard way into a pair of kink systems, centered at $\theta = \pm \ln(1/r)$. We focus on one of them by replacing θ by $\theta - \ln(1/r)$, and keeping this variable finite as the limit is taken. The exponential term in (2.7) is replaced by $-\frac{i}{2}e^\theta$, while to leading order the exponential in (2.8) can be neglected. Furthermore, the singularities in $\ln(1 + e^{\pm f_L(\theta)})$ previously found on the real axis are pushed to $-\infty$, allowing integration contours in any integral involving f_L to be moved across the real θ axis at will. It will be convenient to shift \mathcal{C}_2 down to coincide with \mathcal{C}_1 in all such integrals, and to take \mathcal{C}_1 to be the line $\Im m \theta = -\eta$, with η a suitably-small real number.

For $f_R(\theta)$, the singularities do not disappear but using the reality properties of f_R we can at least rewrite the integrations above and below the axis in terms of the imaginary part of a single integration along \mathcal{C}_1 . The final equation is

$$f_R(\theta) = -\frac{i}{2}e^\theta + i\pi\alpha' + 2i \int_{-\infty}^{\infty} \phi(\theta - (\theta' - i\eta)) \Im m(\ln(1 + e^{f_R(\theta' - i\eta)})) d\theta' \\ - \int_{-\infty}^{\infty} \chi(\theta - (\theta' - i\eta)) f_L(\theta' - i\eta) d\theta' \quad (2.11)$$

$$f_L(\theta) = -i\pi\alpha' + \int_{-\infty}^{\infty} \phi(\theta - (\theta' + i\eta)) f_L(\theta' + i\eta) d\theta' \\ - 2i \int_{-\infty}^{\infty} \chi(\theta - (\theta' - i\eta)) \Im m(\ln(1 + e^{f_R(\theta' - i\eta)})) d\theta' . \quad (2.12)$$

Taking the Fourier transform of (2.12) yields

$$(1 - \tilde{\phi}(k)) \tilde{f}_L(k) = -2i\pi^2\alpha'\delta(k) - 2i\tilde{\chi}(k)\tilde{L}_R(k) \quad (2.13)$$

where $\tilde{L}_R(k)$ denotes the transform of $\Im m(\ln(1 + e^{f_R(\theta' - i\eta)}))$ and our convention for the Fourier transform of $f(\theta)$ is

$$\mathcal{F}[f(\theta)] = \int_{-\infty}^{\infty} f(\theta) e^{-i\theta k} d\theta = \tilde{f}(k) \quad (2.14)$$

with corresponding inverse

$$\mathcal{F}^{-1}[\tilde{f}(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{i\theta k} dk = f(\theta) . \quad (2.15)$$

Inserting (2.13) into the F.T. of (2.11) (first taking $-\frac{i}{2}e^\theta$ to the left hand side to ensure the existence of the transform), we obtain

$$\mathcal{F}[f_R(\theta) + \frac{i}{2}e^\theta] = 2i\pi^2\alpha'\delta(k) \left(1 + \frac{\tilde{\chi}(k)}{1 - \tilde{\phi}(k)}\right) - 2i\tilde{L}_R(k) \left(\tilde{\phi}(k) + \frac{\tilde{\chi}(k)^2}{1 - \tilde{\phi}(k)}\right) . \quad (2.16)$$

This can be compared with the Fourier transform of the kink limit of (2.1), which is

$$\mathcal{F}[f(\theta) - \frac{i}{2}e^\theta] = 2i\pi^2\alpha\delta(k) - 2i\tilde{L}_R(k)\tilde{\varphi}(k) . \quad (2.17)$$

The two will match if

$$\tilde{\varphi}(k) = \tilde{\phi}(k) + \frac{\tilde{\chi}(k)^2}{1 - \tilde{\phi}(k)} \quad (2.18)$$

and

$$\alpha = \alpha' \left(1 + \frac{\tilde{\chi}(0)}{1 - \tilde{\phi}(0)} \right) . \quad (2.19)$$

After some elementary manipulations we can invert the Fourier transform of χ , resulting in

$$\chi(\theta) = - \int_{-\infty}^{\infty} \frac{e^{ik\theta} \sinh(\frac{\pi k}{3}) \cosh(\frac{\pi k}{6})}{\cosh(\frac{\pi k}{2}) \sinh(\frac{\pi k}{3}(\xi - 1))} \frac{dk}{2\pi} \quad (2.20)$$

and, from (2.19),

$$\alpha' = \alpha \frac{\xi}{(\xi - 1)} . \quad (2.21)$$

Since (2.18) involves only $\chi(k)^2$ we have made a choice of sign when taking the square root. We took the negative sign since only then do the values taken by $c_{\text{eff}}(\infty)$ when flowing from a minimal model $\mathcal{M}_{p,q}$ always assume the form $1 - 6/p'q'$, as has to be the case if the infrared destination of the flow is to be another minimal model. It is possible that the flows found for the other sign choice also have an interpretation in some other context, but we will not explore this question here.

3 The flows

By construction, the new massless system exhibits the same ultraviolet central charge as the corresponding massive system:

$$c_{\text{eff}}(0) = 1 - \frac{3\xi}{(\xi+1)}\alpha^2 . \quad (3.1)$$

Furthermore, the fact that the kink limit of the new system can be mapped exactly onto that of the massive system makes it natural to suppose that the recipe for choosing ξ and α given by (2.4), (2.5) and (2.6) also holds for the massless perturbations. We take this as a working hypothesis for now; in the next section it will be subject to some more detailed checks. However, one immediate difficulty should be mentioned: with our convention that the coprime pair (p, q) specifying $\mathcal{M}_{p,q}$ has $p < q$, the value assigned to ξ by (2.4) for ϕ_{12} perturbations is less than 1. The same is true for ϕ_{15} perturbations of $\mathcal{M}_{p,q}$ whenever $q > 4p$. This is not a problem for the massive perturbations, but the massless kernel $\phi(\theta)$ has a pole at $\theta = \frac{2}{3}(\xi-1)\pi i$, which crosses the real θ axis as ξ dips below 1. In addition, the formula (2.21) clearly has a pole at $\xi=1$. All of this

leads to technical complications, and, while they could perhaps be overcome by analytic continuation, we will leave the issue to one side for now. In the rest of this paper we will focus solely on the ϕ_{21} and ϕ_{15} perturbations, found via equations (2.5) and (2.6) respectively, and restrict the ϕ_{15} perturbations to models $\mathcal{M}_{p,q}$ with $q < 4p$.

In the infrared, the massless system mimics a massive system with ξ replaced by $\xi-1$ and α replaced by $\alpha' = \alpha \xi / (\xi-1)$, and so

$$c_{\text{eff}}(\infty) = 1 - \frac{3(\xi-1)}{\xi}(\alpha')^2 = 1 - \frac{3\xi}{(\xi-1)}\alpha^2. \quad (3.2)$$

Consider first the perturbation of $\mathcal{M}_{p,q}$ by ϕ_{21} . For it to be relevant, we must impose that $p > q/2$. Substituting the requisite values of ξ and α (taken from (2.5)) into (3.1) and (3.2) we find:

$$\mathcal{M}_{p,q} + \phi_{21} : \quad p > q/2, \quad c_{\text{eff}}(0) = 1 - \frac{6}{pq}, \quad c_{\text{eff}}(\infty) = 1 - \frac{6}{(q-p)q}. \quad (3.3)$$

Similarly, for the ϕ_{15} perturbations we have

$$\mathcal{M}_{p,q} + \phi_{15} : \quad p < q/2, \quad c_{\text{eff}}(0) = 1 - \frac{6}{pq}, \quad c_{\text{eff}}(\infty) = 1 - \frac{6}{p(4p-q)}. \quad (3.4)$$

As promised at the end of the last section, the infrared limiting values of the effective central charges are all consistent with the destinations of the flows being minimal models[§]. Unfortunately, a knowledge of c_{eff} alone is generally not enough to identify a nonunitary minimal model uniquely, so the destinations cannot be completely pinned down by this information. Moreover, there are cases in which even the knowledge of the full $c_{\text{eff}}(r)$ is not sufficient to identify the model or models involved. The possibility of such an ambiguity was first noted, for massive perturbations, in [37]. It goes by the name of the ‘type II conjecture’, and explicitly it equates the effective central charges of the following models:

$$\mathcal{M}_{p,q} + \phi_{15} \leftrightarrow \mathcal{M}_{q/2,2p} + \phi_{21} \leftrightarrow \mathcal{M}_{2p,q/2} + \phi_{12} \quad (p = 2n+1, q = 2m). \quad (3.5)$$

The second equivalence is just a question of labelling, but the first is much less trivial. However, all follow from the massive NLIE (2.1) and the recipe (2.4)–(2.6), assuming the correctness of these equations (see [25]). Clearly the massless theories, whose NLIEs are parametrised according to essentially the same recipe, will suffer similar ambiguities; some examples will be encountered later.

In spite of these provisos, the forms taken by the results (3.3) and (3.4) make for some obvious conjectures, and these will receive further support, both analytical and numerical, later in the paper. Such checks will never resolve type II ambiguities, so for these cases the conjectures are better motivated by the belief that the pattern observed

[§]had the opposite sign choice for $\chi(\theta)$ been taken in (2.20), the values of $c_{\text{eff}}(\infty)$ for the ϕ_{21} and ϕ_{15} perturbations would have been $1 - 6(q-p)/(qp^2)$ and $1 - 6(4p-q)/(pq^2)$ respectively

in other cases should hold in complete generality. Taking into account the ‘ $p < q$ ’ labeling convention for a minimal model $\mathcal{M}_{p,q}$, the ϕ_{15} case splits into two and we have the following predictions for massless perturbations of general minimal models:

$$\mathcal{M}_{p,q} + \phi_{21} \rightarrow \mathcal{M}_{q-p,q} \quad (p < q < 2p) , \quad (3.6)$$

$$\mathcal{M}_{p,q} + \phi_{15} \rightarrow \mathcal{M}_{p,4p-q} \quad (2p < q < 3p) , \quad (3.7)$$

$$\mathcal{M}_{p,q} + \phi_{15} \rightarrow \mathcal{M}_{4p-q,p} \quad (3p < q < 4p) . \quad (3.8)$$

There is only a single flow on this list from any given minimal model: when ϕ_{21} is relevant, ϕ_{15} is irrelevant, and vice versa. Note also that the case (3.8) can only occur as the last member of a sequence: for $3p < q < 4p$, ξ lies between 1 and 2 and the next step would therefore involve a value of ξ less than 1, and we have already decided to exclude such cases.

The previously-known flows (1.1) and (1.2) are reproduced on setting $q = 2p - 1$ in (3.6) and $q = 2p + 1$ in (3.7). In order to understand the more general pattern, it is convenient to define an ‘index’ $I = 2p - q$, and to rephrase (3.6) and (3.7) as

$$\mathcal{M}_{p,2p-I} + \phi_{21} \rightarrow \mathcal{M}_{p-I,2p-I} , \quad (\xi, \alpha') = \left(\frac{2p}{I}-1, \frac{1}{p-I}\right) , \quad (3.9)$$

$$\mathcal{M}_{p,2p+I} + \phi_{15} \rightarrow \mathcal{M}_{p,2p-I} , \quad (\xi, \alpha') = \left(\frac{2p}{I}, \frac{2}{2p-I}\right) . \quad (3.10)$$

For reference we have included the values of ξ and α' that should be used in the NLIE in order to dial up the corresponding flow. In all cases $I(\text{IR}) = -I(\text{UV})$; the flows (1.1) and (1.2) make up the unique sequence with $|I| = 1$. For $|I| > 1$, there may be more than one sequence, with flows sharing the same value of $|I|$ interlacing each other. This is reminiscent of the generalised staircase models studied in [34], but is a little more complicated owing to the constraint that the pair of integers labelling a minimal model must always be coprime. The number of different sequences with index $\pm I$ is therefore given by the Euler φ -function $\varphi(|I|)$, equal to the number of integers less than $|I|$ which are coprime to $|I|$ (so for $n = 1 \dots 6$, $\varphi(n) = 1, 1, 2, 2, 4, 2$). The index measures the distance from the line $q = 2p$ across which the relevance and irrelevance of the fields ϕ_{21} and ϕ_{15} swap over. All of this is perhaps best seen pictorially, and in figure 1 some of the predicted flows are plotted, superimposed on a grid of the minimal models $\mathcal{M}_{p,q}$. The horizontal arrows of length $|I|$ correspond to ϕ_{21} perturbations, while the vertical arrows, of length $2|I|$, are ϕ_{15} perturbations.

As a first check on our results, we evaluated the effective central charge numerically for a number of members of the $|I| = 1$ series, and made a comparison with the results from the massless TBA equations discussed in [19]. In [19] such equations were written in a ‘universal’ form of the kind first described in [35], but for numerical work it is more convenient to write the equations in the following way:

$$\varepsilon_a(\theta) = \nu_a(\theta) - \sum_{b=1}^n (l_{ab}^{(A_n)} - \delta_{ab}) \int_{-\infty}^{\infty} \mathcal{K}(\theta - \theta') \ln(1 + e^{-\varepsilon_b(\theta')}) d\theta' , \quad (3.11)$$

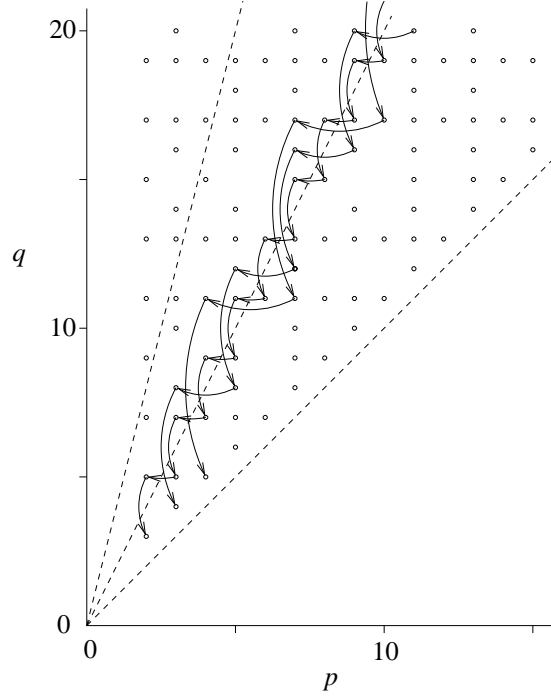


Figure 1: The grid of minimal models $\mathcal{M}_{p,q}$ and a selection of the predicted flows, showing the unique sequences with $|I| = 1$ and $|I| = 2$, and one of the two sequences with $|I| = 3$. Also shown are the lines $q=p$, $q=2p$ and $q=4p$.

$$c_{\text{eff}}(r) = \frac{3}{\pi^2} \sum_{a=1}^n \int_{-\infty}^{\infty} \nu_a(\theta) \ln(1 + e^{-\varepsilon_a(\theta)}) d\theta \quad (3.12)$$

where $n \equiv \xi - 3$ is integer, $l_{ab}^{(A_n)}$ is the incidence matrix of the A_n Dynkin diagram,

$$\nu_a(\theta) = \frac{r}{2} (e^\theta \delta_{a,1} + e^{-\theta} \delta_{a,n}) \quad , \quad (3.13)$$

and

$$\mathcal{K}(\theta) = \frac{\sqrt{3} \sinh 2\theta}{\pi \sinh 3\theta} \quad . \quad (3.14)$$

The agreement between the NLIE and the TBA is extremely good, and is illustrated in table 1 and figure 2.

Satisfied that our NLIE correctly matches the previously known flows, we can turn to the new families with index $|I| > 1$. Figure 3 shows the (unique) series with $|I| = 2$, and figure 4 one of the two possible series with $|I| = 3$. For all of these flows the effective central charge initially increases from its UV value, oscillates as the system

Model	(ξ, α')	r	TBA	NLIE
$\mathcal{M}_{3,7} + \phi_{15}$	$(6, \frac{2}{5})$	0.01	0.71421869185983	0.71421869185981
		0.02	0.71408433022934	0.71408433022935
$\mathcal{M}_{3,5} + \phi_{21}$	$(5, \frac{1}{2})$	0.01	0.59996121217228	0.59996121217225
		0.02	0.59986582853966	0.59986582853967

Table 1: Comparison of c_{eff} calculated using the TBA equations and the NLIE.

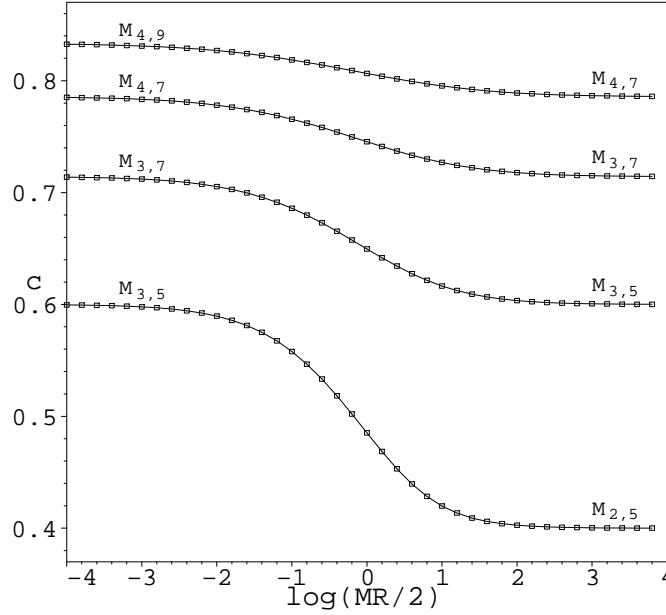


Figure 2: TBA (boxes) and NLIE data (solid lines) for the ‘diagonal’ sequence with $|I| = 1$.

size is increased, before finally flowing to the predicted IR fixed point. This perhaps-surprising behaviour is in contrast to the $|I| = 1$ series where all of the flows are monotonic. We will return to this point at the end of section 4.

Note that no TBA equations are known for the massless flows with $|I| > 1$. This puzzle can be highlighted by looking at some cases where the corresponding massive TBA system *can* be conjectured.

We consider three families of massive systems found in the ‘ADET’ class of models classified in [13]. The first set is obtained from (3.11) simply by replacing the driving term (3.13) by

$$\nu_a(\theta) = r\delta_{a,1} \cosh \theta . \quad (3.15)$$

This gives the massive flows corresponding to the $|I| = 1$ massless flows. For the second set, we continue to use the driving term (3.15), but replace $l_{ab}^{(A_n)}$ with $l_{ab}^{(T_n)}$ (the

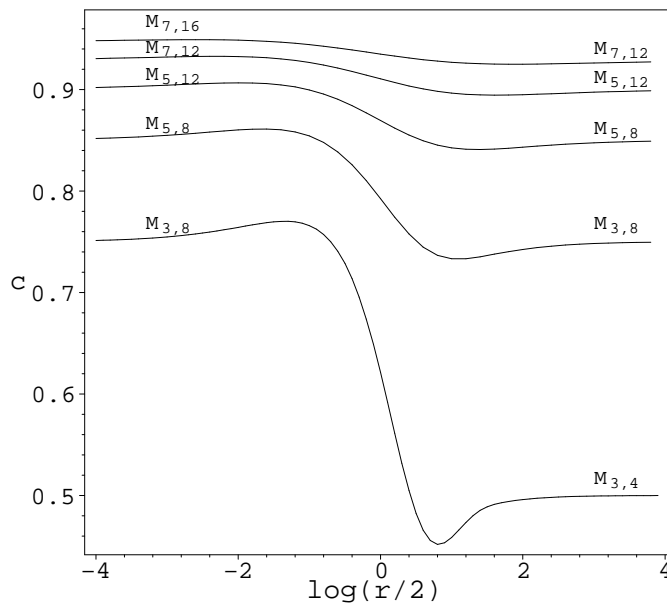


Figure 3: The sequence of nonunitary flows with $|I| = 2$.

incidence matrix of the ‘tadpole’ graph $T_n = A_{2n}/\mathbb{Z}_2$, letting 1 label the node furthest from the ‘tadpole’ node (see figure 5). In [36] these systems were identified with the models $\mathcal{M}_{n+2,2n+2} + \phi_{21}$ for n odd and $\mathcal{M}_{n+1,2n+4} + \phi_{15}$ for n even. These are precisely the models in the $|I| = 2$ series. The massless NLIE predicts, in addition to these previously-known massive flows, the existence of interpolating flows ($n \rightarrow n-1$) within this family.

The final family of massive TBA equations is obtained by replacing the A_n incidence matrix with the D_n one. For these cases the type II conjecture mentioned above plays a rôle, and for each TBA system there are two possible identifications:

$$A_n^+ \equiv \mathcal{M}_{2n-1,2+4n} + \phi_{15} \quad \text{and} \quad A_n^- \equiv \mathcal{M}_{1+2n,4n-2} + \phi_{21} . \quad (3.16)$$

The sets A_n^\pm correspond, in the ultraviolet, to the two series of models with index $|I| = 4$. Turning to the massless flows implied by the NLIE, there is a further ambiguity in the infrared destinations. However, the general pattern of (3.6), (3.7) and (4.6)–(4.8) suggests

$$A_n^\pm \rightarrow A_{n-1}^\mp . \quad (3.17)$$

We have checked the low-lying members of each family of massive TBA equations against the massive NLIE (2.1), finding agreement to our numerical accuracy (about 14 digits) in each case. However, returning to the question of finding massless versions of these equations, we observe a key difference between the first family and the other two: as is clear from figure 5, the graphs for the $|I| = 1$ systems have a \mathbb{Z}_2 symmetry which is in general absent from those for $|I| = 2$ or 4 (the single exception occurs when $|I| = 4$,

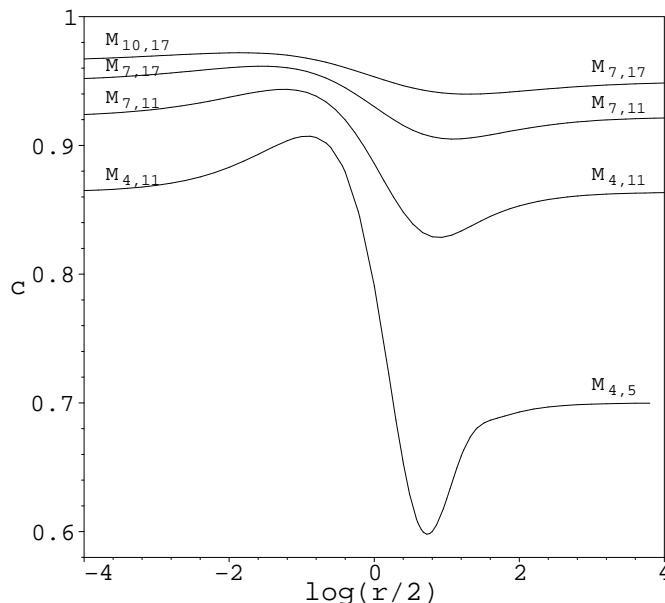


Figure 4: One of the two sequences with $|I| = 3$.

$n = 4$). The trick that allowed us to move between massive and massless systems for $|I| = 1$ by swapping the ν 's defined in (3.15) for those defined in (3.13) relied completely on this symmetry. This ' \mathbb{Z}_2 trick' was first employed in [4]; more generally, the nodes '1' and ' n ' could be replaced by any pair of nodes related by a \mathbb{Z}_2 graph symmetry. To the best of our knowledge, all massless TBA systems that have been discovered to date are related to massive systems in essentially this way. Thus if the T_n and D_n TBA systems do have associated massless versions, they are likely to be of a somewhat different nature to all previously-encountered examples.

The $|I| = 4$, $n = 4$ system might appear to offer a counterexample, since the exceptional symmetry of its graph does allow a massless version of the massive TBA to be constructed. However this equation turns out not to reproduce the massless NLIE. For example, for the massless TBA $c_{\text{eff}}(\infty) = 5/7$, which identifies the IR destination of the flow as $\mathcal{M}_{3,7}$ rather than the $\mathcal{M}_{7,10}$ or $\mathcal{M}_{5,14}$ found by the NLIE. Furthermore, the short-distance expansion of $c_{\text{eff}}(r)$ turns out to be inconsistent with the massless TBA flow being related to the massive one by an analytic continuation of the coupling λ – rather, it is the flow produced by the massless NLIE which has this property. This failure of the \mathbb{Z}_2 trick to produce the analytically-continued massless flow can be put into a more general context. Recall that TBA equations have associated Y-systems, and that these entail a periodicity $Y_a(\theta) = Y_a(\theta + iP)$ for certain functions $Y_a(\theta)$, where P depends on the particular Y-system (see [35] and also [13]). Suppose that a diagram symmetry relates nodes a and \tilde{a} for this system. Then it can be argued that the associated massive and massless TBA equations will be related by analytic continuation

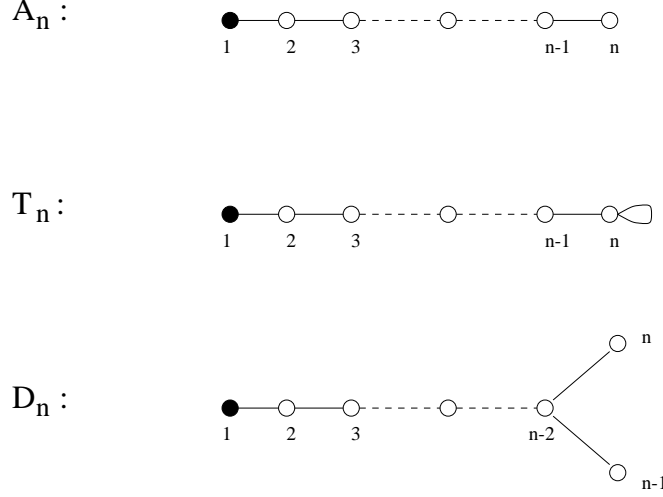


Figure 5: The A_n , T_n and D_n graphs, indicating in each case the node to which the driving term $r \cosh \theta$ in the massive TBA system should be attached.

if $Y_a(\theta) = Y_{\tilde{a}}(\theta + iP/2)$, and not otherwise. In particular, for systems related to the ADET diagrams, such a property holds if and only if the nodes a and \tilde{a} are ‘conjugate’, where conjugation acts on TBA diagrams in the same way as charge conjugation acts on the particles in an affine Toda field theory [38,39]. For the D_n diagrams, the fork nodes are related by charge conjugation for n odd, but not for n even. This thus matches the observation above that the massive and massless D_4 -related systems are not related by continuation in λ ; it also helps to understand from the TBA point of view why the models $H_N^{(\pi)}$ of [8] are related to $H_N^{(0)}$ by analytic continuation for N odd, but not for N even.

We conclude this section with some further observations. Following [36] we note that two models in the $|I| = 2$ set possess $N=1$ supersymmetry in the ultraviolet. In the notation of [36] we have

$$S\mathcal{M}_{2,8} + \hat{\phi}_{13}^{top} \simeq \mathcal{M}_{3,8} + \phi_{15} , \quad (3.18)$$

and

$$S\mathcal{M}_{3,7} + \hat{\phi}_{15}^{top} \simeq \mathcal{M}_{7,12} + \phi_{21} . \quad (3.19)$$

The operators $\hat{\phi}_{13}^{top}$ and $\hat{\phi}_{15}^{top}$ are SUSY-preserving, so the massless flows $\mathcal{M}_{3,8} \rightarrow \mathcal{M}_{3,4}$ and $\mathcal{M}_{7,12} \rightarrow \mathcal{M}_{5,12}$ should exhibit spontaneous breaking of $N=1$ SUSY. The situation is reminiscent of the flow between the tricritical Ising and Ising models discussed in [4, 40]. In particular, the flow $\mathcal{M}_{3,8} \rightarrow \mathcal{M}_{3,4}$ is to a theory of a single massless free Majorana fermion, just as was the case in [4, 40]. This particle, in turn, was identified in [4, 40] as the massless Goldstone fermion. Finally, we mention that the massive TBA system for $\mathcal{M}_{3,8} + \phi_{15}$ was alternatively obtained in [36] as a folding of the $N=2$

supersymmetric model with spontaneously-broken \mathbb{Z}_k symmetry of [41], for $k=3$. This suggests that there should also be a (probably nonunitary) flow from the ‘parent’ $N=2$ supersymmetric theory into a massless free fermionic theory. Given that a similar interpolating phenomenon is also present for $k=2$ [¶] it is natural to suppose that the same behaviour will be exhibited for all of the \mathbb{Z}_k -related models of [41]. It would be interesting to check this conjecture using the $a_{k-1}^{(1)}$ -generalisations of our NLIE.

4 Perturbation theory

The claims of the last section can be put on a more secure footing by taking a closer look at the behaviour of the effective central charges at small and large r . For this we will need the leading asymptotics of the kernels $\phi(\theta)$ and $\chi(\theta)$ as $\theta \rightarrow -\infty$. That of ϕ can be found from the residues of the poles in the integrand of (2.10) at $k = -i$ and $k = -3i/(\xi-1)$, and is:

$$\phi(\theta) \sim -\frac{\sqrt{3}\sin(\frac{\pi}{3}\xi)}{\pi\sin(\frac{\pi}{3}(\xi-1))}e^\theta + \frac{3\sin(\frac{\pi}{(\xi-1)})\cos(\frac{\pi}{2(\xi-1)})}{\pi(\xi-1)\cos(\frac{3\pi}{2(\xi-1)})}e^{3\theta/(\xi-1)} + \dots, \quad \theta \rightarrow -\infty. \quad (4.1)$$

Similarly,

$$\chi(\theta) \sim -\frac{3}{2\pi\sin(\frac{\pi}{3}(\xi-1))}e^\theta - \frac{3\sin(\frac{\pi}{(\xi-1)})\cos(\frac{\pi}{2(\xi-1)})}{\pi(\xi-1)\cos(\frac{3\pi}{2(\xi-1)})}e^{3\theta/(\xi-1)} + \dots, \quad \theta \rightarrow -\infty. \quad (4.2)$$

We first analyse the NLIE as $r \rightarrow \infty$, to see whether its behaviour is compatible with the claimed infrared destinations of the massless flows. In spite of the fact that conformal perturbation theory about the infrared fixed point is not renormalisable, there are a number of unambiguous predictions against which the equation can be checked. The key ideas are set out in [4, 43], and are further discussed in, for example, [9, 16, 17].

In general, the infrared model will be described by an action of the form

$$S = S_{\text{IR}}^* + \mu_1 \int \psi d^2x + \mu_2 \int T\bar{T} d^2x + (\text{further terms}) \quad (4.3)$$

where S_{IR}^* is the action, and ψ one of the (irrelevant) primary fields, of the infrared conformal field theory. ψ might be absent, in which case the only operators attracting the flow to the IR fixed point would be the descendents of the identity, $T\bar{T}$ being the least irrelevant example. On dimensional grounds, the couplings μ_1 and μ_2 are related to the single crossover scale M as

$$\mu_1 = \kappa_1 M^{2-2h}, \quad \mu_2 = \kappa_2 M^{-2}, \quad (4.4)$$

with κ_1 and κ_2 dimensionless constants and h the conformal dimension of ψ .

[¶]for more on the $k=2$ case, and its connection with the theory of dense polymers, see [20, 42]

Standard methods of perturbed conformal field theory can now be used to calculate the first corrections to $c_{\text{eff}}(\infty)$, with results that are good at least up to the order at which the ‘further terms’ cut in. The first perturbing term in (4.3) yields a series in $\mu_1 R^{2-2h} = \kappa_1 r^{2-2h}$ if all powers of μ_1 contribute, or $\mu_1^2 R^{4-4h} = \kappa_1^2 r^{4-4h}$ if only even powers appear, as happens when ψ is odd under some symmetry of S_{IR}^* . The second term always contributes a series in $\mu_2 R^{-2} = \kappa_2 r^{-2}$, the first two terms of which were found explicitly in [43]. Putting everything together gives $c_{\text{eff}}(r)$ the following large- r expansion:

$$c_{\text{eff}}(r) \sim c_{\text{eff}}(\infty) + (\text{a series in } \kappa_1 r^{2-2h} \text{ or } \kappa_1^2 r^{4-4h}) \\ - \frac{\pi^3 c_{\text{eff}}(\infty)^2}{6} \kappa_2 r^{-2} + \frac{\pi^6 c_{\text{eff}}(\infty)^3}{18} \kappa_2^2 r^{-4} + (\text{further terms}) . \quad (4.5)$$

In contrast to the UV situation to be discussed shortly, this series is only expected to be asymptotic, and furthermore there is very little control over the omitted ‘further terms’. Nevertheless, it allows for some useful comparisons with results from the NLIE.

When r is large, the nontrivial behaviours of the functions f_L and f_R appearing in (2.7) and (2.8) are concentrated in ‘kink’ regions near $\theta = -\ln r$ and $\theta = \ln r$ respectively. So long as the contours \mathcal{C}_1 and \mathcal{C}_2 are kept a finite distance from the real θ -axis, the functions $\ln(1+e^{\pm f_L})$ and $\ln(1+e^{\pm f_R})$ appearing in these equations are doubly-exponentially suppressed in the central zone between the kinks, and so the principal interaction between equations (2.7) and (2.8), and hence the principal correction to $c_{\text{eff}}(r)$, comes from the exponential tail of the kernel function $\chi(\theta)$, given by (4.2). Since we are only worrying about the first few terms in the expansion, and we are free to vary ξ so as to avoid any ‘resonance’ effects, we can discuss effects of the two terms in (4.2) separately. Consider first the second term, decaying as $e^{3\theta/(\xi-1)}$ as $\theta \rightarrow -\infty$. Inserted into (2.7) and considered iteratively, corrections to $f_R(\theta)$ as a series in $r^{-6/(\xi-1)}$ will be generated. Feeding through into $c_{\text{eff}}(r)$, these corrections can be matched against the κ_1 terms in (4.5), allowing h , the conformal dimension of the field ψ , to be extracted. Comparing with the Kac formula $h_{ab} = ((bp' - aq')^2 - (p' - q')^2)/(4p'q')$ at the appropriate (IR) values of p' and q' leads us to conjecture the following pattern of arriving operators ψ :

$$\mathcal{M}_{p,q} + \phi_{21} \rightarrow \mathcal{M}_{q-p,q} \quad (p < q < 2p) \quad \text{arriving via } \phi_{21} , \quad (4.6)$$

$$\mathcal{M}_{p,q} + \phi_{15} \rightarrow \mathcal{M}_{p,4p-q} \quad (2p < q < 3p) \quad \text{arriving via } \phi_{15} , \quad (4.7)$$

$$\mathcal{M}_{p,q} + \phi_{15} \rightarrow \mathcal{M}_{4p-q,p} \quad (3p < q < 4p) \quad \text{arriving via } \phi_{51} . \quad (4.8)$$

In making these identifications, we took account of the fact that ϕ_{21} is an odd operator, while ϕ_{15} is even.

Exactly the same line of argument starting from the e^θ term in (4.2) reveals a further series of corrections to c_{eff} as powers of r^{-2} , perfectly adapted to match the $T\bar{T}$ terms in (4.5). However this time it is possible to say more, by observing that the effect of this part of $\chi(\theta)$ on equations (2.7) and (2.8) can be reabsorbed into a shift of r by a

constant, and furthermore that this constant can be expressed in terms of $c(r)$. This allows the first few terms of the iterative expansion to be found exactly. The same idea was used in [17] to extract IR asymptotics from various massless TBA systems; in the current context we find

$$c_{\text{eff}}(r) \sim c_{\text{eff}}(\infty) - \frac{\pi c_{\text{eff}}(\infty)^2}{\sin \frac{\pi}{3}(\xi-1)} r^{-2} + \frac{2\pi^2 c_{\text{eff}}(\infty)^3}{\sin^2 \frac{\pi}{3}(\xi-1)} r^{-4} + \dots \quad (4.9)$$

Matching r^{-2} terms in (4.5) and (4.9) gives the exact relation $\kappa_2 = 6/(\pi^2 \sin \frac{\pi}{3}(\xi-1))$; the fact that the r^{-4} terms then agree provides a nontrivial check on the IR behaviour of the massless NLIE.

We also performed some numerical fits on the IR data. Our results are summarised in table 2. Accuracy was fairly low, but where relevant we have also included the predicted (‘exact’) values of coefficients, obtained from equation (4.9).

Model	$c_{\text{eff}}(r) - c_{\text{eff}}(\infty)$
$\mathcal{M}_{3,5} + \phi_{21}$	$0.58041577 r^{-2} + 0.000061 r^{-3} + 1.66 r^{-4} + 0.12 r^{-\frac{9}{2}} + \dots$
Exact	$0.58041579 r^{-2} + 1.6844 r^{-4}$
$\mathcal{M}_{3,7} + \phi_{15}$	$1.30591 r^{-2} + 1.65488 r^{-\frac{12}{5}} + \dots$
Exact	$1.30593 r^{-2}$
$\mathcal{M}_{4,7} + \phi_{21}$	$0.51029 \ln(r) r^{-2} - 0.68683 r^{-2} + \dots$
Exact	$0.51020 \ln(r) r^{-2}$
$\mathcal{M}_{7,11} + \phi_{21}$	$-6.8516 r^{-2} + 8.8213 r^{-\frac{9}{4}} + \dots$
Exact	$-6.8510 r^{-2}$
$\mathcal{M}_{5,8} + \phi_{21}$	$-1.1253 \ln(r) r^{-2} + 1.3757 r^{-2} + \dots$
Exact	$-1.125 \ln(r) r^{-2}$
$\mathcal{M}_{6,17} + \phi_{15}$	$-2.32081 r^{-2} + 12.85 r^{-4} - 1.527 r^{-\frac{30}{7}} + \dots$
Exact	$-2.32082 r^{-2} + 12.56 r^{-4}$

Table 2: Comparison of the infrared expansion of $c_{\text{eff}}(r) - c_{\text{eff}}(\infty)$ for various models against the exact coefficients.

The situation in the ultraviolet is in many respects much simpler. Conformal perturbation theory gives direct access to a function $c_{\text{pert}}(r)$, related to the ground state energy (1.3) as $E_0(M, R) = -\frac{\pi}{6R} c_{\text{pert}}(r)$. Note that c_{pert} contains a bulk part which must be subtracted before comparisons are made with the NLIE. For a theory perturbed by a relevant primary operator ϕ with scaling dimensions $(h_{\text{UV}}, h_{\text{UV}})$, c_{pert} has

the expansion [44, 45]

$$c_{\text{pert}}(r) = c(0) + \sum_{n=1}^{\infty} C_n (\lambda R^y)^n \quad (4.10)$$

and, in contrast to the situation in the IR, the series is expected to have a finite radius of convergence. Here $y = 2 - 2h_{\text{UV}}$, λ is the coupling, and the coefficients C_n are given in terms of the connected correlation functions of the perturbing field on the plane as

$$C_n = \frac{12(-1)^n}{n!(2\pi)^{yn-1}} \int \prod_{j=2}^n \frac{d^2 z_j}{|z_j|^y} \langle V(0) \phi(1, 1) \phi(z_2, \bar{z}_2) \dots \phi(z_n, \bar{z}_n) V(\infty) \rangle_C, \quad (4.11)$$

where V creates the CFT ground state on the cylinder. This is the state with lowest conformal dimension, and for a general minimal model $\mathcal{M}_{p,q}$ it corresponds to the field $\phi_0 \equiv \phi_{ab}$ with a and b integers satisfying $bp - aq = 1$. (Only for the unitary models $\mathcal{M}_{p,p+1}$ does it coincide with the conformal vacuum ϕ_{11} .)

For the perturbing operator, we will be interested in $\phi = \phi_{15}$ and $\phi = \phi_{21}$. As mentioned above, ϕ_{21} is odd, so only the even coefficients C_{2n} are nonzero for this case. Since λ must be related to the crossover scale M as

$$\lambda = \kappa M^y \quad (4.12)$$

with κ a dimensionless constant, (4.10) is a series in r^y for the ϕ_{15} perturbations, and r^{2y} for the ϕ_{21} perturbations.

The effective central charge calculated using the NLIE is expected to expand as

$$c_{\text{eff}}(r) = c_{\text{eff}}(0) + B(r) + \sum_{n=1}^{\infty} c_n r^{yn} \quad (4.13)$$

for some value of y . Periodicity arguments suggest that this will be a series in $r^{6/(1+\xi)}$, and if ξ is chosen according to (2.5) or (2.6) then $y = 2 - 2h_{\text{UV}}$ and the perturbative expansion (4.10) is matched, with h_{UV} the conformal dimension of either ϕ_{21} or ϕ_{15} , and all odd terms zero for the ϕ_{21} case [25]. The irregular bulk term $B(r)$ must be subtracted before c_{pert} can be compared with c_{eff} . Fortunately, it can be obtained exactly from the NLIE, using a small generalisation of arguments used in [44] and [21]. The term we need is given by the behaviour as $r \rightarrow 0$ of

$$2 \frac{3ir}{2\pi^2} \left[\int_{c_2^2} e^{-\theta} \frac{d}{d\theta} \ln(1 + e^{f_L(\theta)}) d\theta - \int_{c_1^1} e^{-\theta} \frac{d}{d\theta} \ln(1 + e^{-f_L(\theta)}) d\theta \right], \quad (4.14)$$

where the ‘ > 0 ’ indicates that only those parts of the contours \mathcal{C}_1 and \mathcal{C}_2 with positive real part should be taken, and the symmetry between f_L and f_R was used to trade the first two integrals in (2.9) for the prefactor 2. Consider the $r \rightarrow 0$ limit of (2.7) and (2.8) in the region $0 \ll \Re \theta \ll \ln(1/r)$, where the driving term $-i\frac{r}{2}e^{-\theta}$ in (2.8) can be dropped. Take the derivative of these equations with respect to θ , and extract the

contributions to the convolutions proportional to e^θ using (4.1) and (4.2). These should cancel either against the remaining driving term or between themselves to ensure that the functions f_L and f_R have no such dependency. This leads to the equations

$$\begin{aligned}
i\frac{r}{2} &= -\frac{\sqrt{3}\sin(\frac{\pi}{3}\xi)}{\pi\sin(\frac{\pi}{3}(\xi-1))}\left[\int_{c_1>0}^{c_2} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{f_R(\theta')})d\theta' - \int_{c_2>0}^{c_1} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{-f_R(\theta')})d\theta'\right] \\
&\quad -\frac{3}{2\pi\sin(\frac{\pi}{3}(\xi-1))}\left[\int_{c_1>0}^{c_2} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{-f_L(\theta')})d\theta' - \int_{c_2>0}^{c_1} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{f_L(\theta')})d\theta'\right]; \\
0 &= -\frac{\sqrt{3}\sin(\frac{\pi}{3}\xi)}{\pi\sin(\frac{\pi}{3}(\xi-1))}\left[\int_{c_2>0}^{c_1} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{f_L(\theta')})d\theta' - \int_{c_1>0}^{c_2} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{-f_L(\theta')})d\theta'\right] \\
&\quad -\frac{3}{2\pi\sin(\frac{\pi}{3}(\xi-1))}\left[\int_{c_2>0}^{c_1} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{-f_R(\theta')})d\theta' - \int_{c_1>0}^{c_2} e^{-\theta'}\frac{d}{d\theta'}\ln(1+e^{f_R(\theta')})d\theta'\right].
\end{aligned}$$

Solving for the integrals needed for the evaluation of (4.14), we find

$$B_{massless}(r) = -\frac{3}{\pi}\frac{\sin(\frac{\pi}{3}\xi)\sin(\frac{\pi}{3}(\xi-1))}{\sin(\pi\xi)}r^2. \quad (4.15)$$

A similar if slightly simpler analysis of the massive equation (2.1) predicts

$$B_{massive}(r) = -\frac{\sqrt{3}\sin(\frac{\pi}{3}\xi)}{2\pi\sin(\frac{\pi}{3}(\xi+1))}r^2. \quad (4.16)$$

It can be checked that this formula combines the results given separately for ϕ_{12} , ϕ_{21} and ϕ_{15} perturbations in [46, 47].

The formula (4.15) has a pole when $\xi + 1$ is an integer multiple of 3. Since the overall result remains finite this infinity should cancel against one of the terms in the regular expansion of c_{pert} , leaving a logarithmic contribution to the final result (see for example [17, 48, 49]). At this point the calculation might appear to split into two cases: the term proportional to r^2 in the perturbative expansion occurs at an order n which depends both on the dimension and on the ‘parity’ of the perturbing operator: $n = 2/y$ for ϕ_{15} , and $1/y$ for ϕ_{21} . However $y = 6/(\xi+1)$ for ϕ_{15} and $3/(\xi+1)$ for ϕ_{21} [25], so $n = (\xi+1)/3$ in both cases and they can be treated simultaneously. The logarithm is found by evaluating

$$\lim_{\xi \rightarrow 3n-1} -\frac{3\sin(\frac{\pi}{3}\xi)\sin(\frac{\pi}{3}(\xi-1))}{\pi\sin(\pi\xi)}\left(r^2 - r^{2n\frac{3}{\xi+1}}\right) \quad (4.17)$$

which yields

$$B_{massless}(r)\Big|_{\xi=3n-1} = (-1)^n \frac{3}{2\pi^2 n} r^2 \ln r. \quad (n \in \mathbb{N}) \quad (4.18)$$

Similarly, and for exactly the same values of ξ , logarithms arise in the massive perturbations. In these cases we find

$$B_{massive}(r)\Big|_{\xi=3n-1} = \frac{3}{2\pi^2 n} r^2 \ln r . \quad (n \in \mathbb{N}) \quad (4.19)$$

This seems to leave the coefficient of r^2 term unknown, but one piece of information can be extracted, which will be used later. Recall that the logarithm is linked to a divergence in the perturbative integral defining the term C_n . Providing the same regularisation scheme is used for both flow directions, the perturbative contributions should cancel upon taking their sum or difference. The remaining finite contribution can be found directly from (4.15) and (4.16):

$$\left(B_{massless}(r) \pm B_{massive}(r)\right)\Big|_{\xi=3n-1} = \pm \frac{\sqrt{3}}{4\pi} r^2 \quad (n \in \mathbb{N}) \quad (4.20)$$

where the plus signs should be chosen if n is odd and the minus signs if n is even.

Using an iterative method to solve (2.7) and (2.8), $c_{\text{eff}}(r)$ can be calculated to high accuracy. After subtracting the ultraviolet central charge and the relevant bulk term, the perturbative coefficients c_n can be estimated via a polynomial fit. To make the comparison with CPT, we must also fix the value of κ . For the massive systems this was determined exactly in [46, 47], and it turns out, numerically at least, that the same relationship between λ and M holds in the massless cases, apart from a factor of either -1 or i . As explained in [19], the relationship between the massive and massless perturbations depends on the parity of the perturbing operator. For ϕ_{15} , the massive behaviour is related to the massless by flipping the sign of the coupling constant λ . This has no effect for the odd operator ϕ_{21} ; instead the required transformation is $\lambda \rightarrow i\lambda$. The CPT coefficients C_n are the same for both flow directions so, provided the mass and the crossover scales M are equal, we expect

$$\phi_{15} : \quad c_n = (-1)^n \tilde{c}_n , \quad (4.21)$$

$$\phi_{21} : \quad c_{2n} = (-1)^n \tilde{c}_{2n} , \quad (4.22)$$

where \tilde{c}_n denotes the expansion coefficients obtained using the massive NLIE. Tables 3 and 4 present the first few perturbative coefficients for the models $\mathcal{M}_{4,11}$ and $\mathcal{M}_{5,8}$ perturbed in both massless and massive directions. The relative signs of the coefficients confirm relations (4.21) and (4.22). Similar results were obtained for the models $\mathcal{M}_{3,8}$, $\mathcal{M}_{5,12}$ and $\mathcal{M}_{3,10}$ perturbed by ϕ_{15} , and the models $\mathcal{M}_{3,4}$, $\mathcal{M}_{3,5}$ and $\mathcal{M}_{7,11}$ perturbed by ϕ_{21} . These provide strong support for our claim that, modulo the bulk terms, the results from the massive and massless integral equations are related by analytic continuation.

For the ϕ_{15} perturbations, it is also simple to check the value of κ directly, since the first term in the perturbative expansion (4.10) is just given by

$$C_1 = -12 (2\pi)^{(1-y)} C_{\phi_0 \phi_{15} \phi_0} \quad (4.23)$$

n	\tilde{c}_n (massive)	c_n (massless)	Δc_n
0	19/22	19/22	/
1	-0.393148695201	0.393148695201	$\pm 7 \cdot 10^{-13}$
2	0.0166115829	0.0166115834	$\pm 5 \cdot 10^{-10}$
3	-0.001225836	0.001225827	$\pm 8 \cdot 10^{-9}$
4	-0.00003425	-0.00003422	$\pm 2 \cdot 10^{-8}$
5	0.0000116	-0.000012	$\pm 3 \cdot 10^{-7}$

Table 3: Comparison of massive and massless UV coefficients for $\mathcal{M}_{4,11} + \phi_{15}$. The last column reports the difference between the absolute values of the two.

n	\tilde{c}_n (massive)	c_n (massless)	Δc_n
0	0.85	0.85	/
2	-0.1128538069088	0.1128538069085	$\pm 3 \cdot 10^{-13}$
4	0.1607667041	0.1607667048	$\pm 7 \cdot 10^{-10}$
6	0.003799534	-0.003799535	$\pm 1 \cdot 10^{-9}$
8	-0.0009586	-0.0009585	$\pm 1 \cdot 10^{-7}$
10	-0.0001554	0.0001554	$\pm 8 \cdot 10^{-8}$

Table 4: Comparison of massive and massless UV coefficients for $\mathcal{M}_{5,8} + \phi_{21}$. The last column reports the difference between the absolute values of the two.

Model	κ_{num}^2	κ^2
$\mathcal{M}_{5,12}$	0.012665147953	0.012665147955 ...
$\mathcal{M}_{7,16}$	0.0247653386712	0.0247653469711 ...
$\mathcal{M}_{4,11}$	-0.06407485530	-0.06407485531 ...
$\mathcal{M}_{7,17}$	0.0092588732986	0.0092588732985 ...
$\mathcal{M}_{10,23}$	0.02340788397	0.0234078837 ...
$\mathcal{M}_{3,10}$	-0.0188367	-0.0188368 ...

Table 5: Comparison of exact and numerical values of κ^2 for ϕ_{15} perturbed models.

where $C_{\phi_0 \phi_{15} \phi_0}$, the operator product coefficient between the perturbing operator ϕ_{15} and the ground state ϕ_0 , can be found in [50]. Comparing (4.10) and (4.13) at order $n = 1$, we have

$$\kappa = \frac{c_1}{C_1} . \quad (4.24)$$

Using this formula and the massless NLIE, we estimated κ^2 numerically for a number of models. Table 5 reports the results, and compares them with the exact expression for κ^2 for (massive) ϕ_{15} perturbations given in [47]:

$$\lambda^2 = \frac{4^2 (\xi - 1)^2 \gamma(\frac{2+\xi}{2(1+\xi)}) \gamma(\frac{5\xi}{2(1+\xi)})}{\pi^2 (2 - 3\xi)^2 (2 - \xi)^2 \gamma^2(\frac{3+\xi}{1+\xi})} \left[\frac{M \Gamma(\frac{\xi+1}{3})}{\sqrt{3} \Gamma(\frac{1}{3}) \Gamma(\frac{\xi}{3})} \right]^{\frac{12}{\xi+1}} \quad (4.25)$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ and M is the mass of the lightest kink.^{||} The agreement is clearly very good. Note that κ^2 is sometimes negative – these are cases where, in the normalisations of [50], $C_{\phi_0 \phi_{15} \phi_0}$ turns out to be purely imaginary.

Now we return to the behaviour of the flows (1.1) and (1.2), to mention one reason why, exceptionally, $c_{\text{eff}}(r)$ should be a monotonic function of r for these flows. The behaviour of a flow is determined at small r by the first nonzero term in the perturbative expansion. Thus whether c_{eff} initially increases or decreases will typically be determined by the sign of $C_1 \lambda$ (ϕ_{15}) or $C_2 \lambda^2$ (ϕ_{21}). In particular, if C_1 (respectively C_2) is nonzero then for one sign of λ (or λ^2), c_{eff} will initially increase, leading to an immediate violation of the ‘ c_{eff} -theorem’, and a flow which must be non-monotonic if $c_{\text{eff}}(\infty)$ is to be less than $c_{\text{eff}}(0)$. But for the model $\mathcal{M}_{3,5} + \phi_{21}$, the first coefficient C_2 was calculated to be zero in [19]. In this case the asymptotic UV behaviour is instead controlled by $C_4 \lambda^4$ and this permits c_{eff} to decrease initially for both (massless and massive) signs of λ^2 . A similar calculation for the other models in the $|I| = 1$ series finds C_1 (ϕ_{15}) or C_2 (ϕ_{21}) to be zero and thus, as in the first model, all of the flows are able to be monotonic. A numerical fit of the data for models higher up in the series confirms these coefficients are zero within our numerical accuracy. For all other sequences of models, such vanishings

^{||} note the relation was given in [47] in terms of $\xi^{\text{FLZZ}} = \frac{p}{q-p}$ and m , the mass of the lightest breather, related to M as $m = 2M \sin(\frac{\pi}{3}\xi)$

of C_1 or C_2 do not occur, forcing at least one of each pair of massive and massless flows to be non-monotonic. One remaining mystery, out of many, is why it should always be the *massless* flow which is non-monotonic.

The $|I| = 1$ sequence of flows was found in [12] via an associated staircase model, and the final step of this staircase interpolates in the infrared to a massive model with $c_{\text{eff}} = 0$. It turns out that the NLIE also reproduces this behaviour: even though the operator ϕ_{15} is not a member of the Kac table for $\mathcal{M}_{2,5}$, (3.7) formally predicts a further flow, to a theory with $c_{\text{eff}} = 0$: $\mathcal{M}_{2,5} + \phi_{15} \rightarrow \mathcal{M}_{2,3}$. There is nothing to stop us using the massless NLIE (2.7,2.8) and the ϕ_{15} recipe (2.6) to compute an effective central charge for this putative flow. Our numerical results show c_{eff} to have an exponential behaviour in the infrared, as would be expected for a massive, rather than a massless, flow. Furthermore, comparing c_{eff} (massless) with that of the massive flow $\mathcal{M}_{2,5} + \phi_{15}$ calculated using (2.1) and (2.6), we find the effective central charges match exactly. They also turn out to coincide with the results from the more standard massive flow $\mathcal{M}_{25} + \phi_{12}$, computed using (2.1) and (2.4). This means that the flow is at least physically reasonable, since ϕ_{12} is the single relevant primary field in $\mathcal{M}_{2,5}$. However, it remains a curiosity that the same flow can be found from three different nonlinear integral equations. A sample of our numerical results is shown in table 6. Note that this example shows that the straightforward prediction of scaling dimensions based on periodicity arguments is not always correct: for $\mathcal{M}_{2,5} + \phi_{15}$ one would have expected a series in $r^{6/5}$ for $c_{\text{eff}}(r)$, whereas in fact the expansion is in powers of $r^{12/5}$.

r	$\mathcal{M}_{2,5} + \phi_{15}$ (massless)	$\mathcal{M}_{2,5} + \phi_{15}$ (massive)	$\mathcal{M}_{2,5} + \phi_{12}$ (massive)
0.01	0.39997512539833	0.39997512539839	0.39997512539863
0.2	0.39254149935036	0.39254149935037	0.39254149935054

Table 6: Comparison of effective central charge for the three a priori different models.

If we compare the infrared destinations of (3.7) and (3.8) we see it is possible for two different ultraviolet models, both perturbed by ϕ_{15} , to flow to the same infrared fixed point, one attracted via the irrelevant operator ϕ_{15} and the other via ϕ_{51} . Figure 6 illustrates two such flows. As was noted in §3, the sequence attracted by ϕ_{51} necessarily stops at this model, but the other sequence may continue to flow down further.

For some models, the predictions (4.6)–(4.8) must be treated with caution, as we were unable to check them explicitly. Examples are the minimal models $\mathcal{M}_{p,q}$ perturbed by ϕ_{15} when p and q are related as $4p - q = 1$. For these cases the twist α' is equal to 2, and a trivial shift of $f_{L/R}(\theta)$ leaves a system with $\alpha' = 0$. This prevents us from tuning in a straightforward way onto the required ultraviolet model: numerically, the massless NLIE simply recovers a flow from $c_{\text{eff}}(0) = 1$ to $c_{\text{eff}}(\infty) = 1$.

We also observe that (3.6) formally predicts the following flows from unitary minimal models:

$$\mathcal{M}_{p,p+1} + \phi_{21} \rightarrow \mathcal{M}_{1,p} , \quad (\xi, \alpha') = \left(\frac{p}{p+1}, 1\right) . \quad (4.26)$$

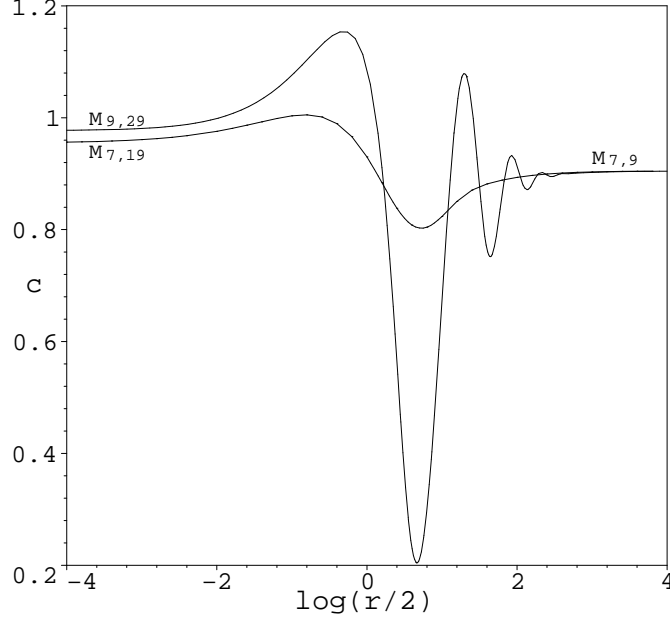


Figure 6: Two different flows to the same IR point, $\mathcal{M}_{7,9}$: $\mathcal{M}_{9,29} + \phi_{15}$ attracted via ϕ_{51} , and $\mathcal{M}_{7,19} + \phi_{15}$ attracted via ϕ_{15} .

However it is a little hard to decide what is meant by $\mathcal{M}_{1,p}$, and indeed, while our numerical results indicate that these flows behave as expected for small r , at some intermediate scale $c_{\text{eff}}(r)$ appears to have a discontinuity. This could be linked to α' being equal to 1. In any event, we suspect there may be a square root singularity though a detailed investigation will have to be left for future work. Note that by the type II equivalence, a similar phenomenon occurs for the models $\mathcal{M}_{p,4p-2}$ perturbed by ϕ_{15} . In one case a detailed check on these exceptional flows can be made with relative ease: the Ising model $\mathcal{M}_{3,4}$ perturbed by ϕ_{21} . For real coupling λ , an exact expression for the (massive) effective central charge was given in [45]. Using this we can make an exact prediction for c_{eff} in the massless case: we send $r \rightarrow ir$ in the perturbative expansion, swap the logarithmic bulk term for that of the massless model (4.15) and find the coefficient of the r^2 term using (4.20). The result is:

$$c_{\text{eff}}(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2} \left[\ln r - \frac{1}{2} - \ln \pi + \gamma_E + \frac{\pi}{2\sqrt{3}} \right] + \frac{6}{\pi} \sum_{k=1}^{\infty} \left(\sqrt{(2k-1)^2 \pi^2 - r^2} - (2k-1)\pi + \frac{r^2}{2(2k-1)\pi} \right) \quad (4.27)$$

where $\gamma_E = 0.57721556\dots$ is the Euler-Mascheroni constant. This formula predicts a square root singularity at $r=\pi$, preventing a smooth interpolating flow to the far infrared, but we can at least match against the NLIE for values of r out to this point. As table 7 shows, our numerical results agree well with the exact formula (4.27).

r	$c_{\text{eff}}^{NLE}(r)$	$c_{\text{eff}}^{\text{exact}}(r)$
0.0001	0.500000014242146	0.500000014242144
0.6	0.535668551455508	0.535668551455511
1.1	0.49940046725090	0.49940046725085
1.4	0.4131296597211	0.4131296597212
1.8	0.18775453205	0.18775453226

Table 7: Comparison of exact and numerical values for c_{eff} for the ϕ_{21} perturbation of $\mathcal{M}_{3,4}$.

5 A remark on ϕ_{13} perturbations

In this short section we comment on some features of massless ϕ_{13} perturbations which are revealed when similar methods are used. In [20] numerical results were only presented for the zero twist ($c_{\text{eff}}(0) = c_{\text{eff}}(\infty) = 1$) case, but the same equation can describe a minimal model $\mathcal{M}_{p,q}$ perturbed by ϕ_{13} . One simply has to set the kernel parameter ζ (p in [20]) to $p/(q-p)$, and the twist α in that paper to π/p .

Defining an index $J = q - p$ [5], $\zeta = p/J$ and the flow from ζ to $\zeta - 1$ is

$$\mathcal{M}_{p,p+J} + \phi_{13} \rightarrow \mathcal{M}_{p-J,p} \quad \text{attracted via } \phi_{31} . \quad (5.1)$$

Again, the number of sequences for each J is given by the Euler φ -function, $\varphi(J)$. As in the $a_2^{(2)}$ case, we cannot access all possible minimal models via this equation. The kernel $\phi(\theta)$ in the massless equation of [20] has a pole at $i\pi(\zeta - 1)$ which crosses the real θ axis as ζ falls below 1. As before, this could probably be overcome using analytic continuation but for now we choose $\zeta > 1$, requiring $2p > q$, to prevent such problems occurring. We also find a set of models, $\mathcal{M}_{p,2p-1} + \phi_{13}$, where $\alpha' = 1$ and the would-be IR model is $\mathcal{M}_{1,p}$. As in the $a_2^{(2)}$ case, c_{eff} suffers some sort of discontinuity at an intermediate scale for these flows.

Solving the massless equation in [20] for the unitary cases ($J = 1$), we find that results from the TBA equations of [4] are matched to high accuracy**. The results for $J > 1$ are perhaps more interesting, since TBA systems for these nonunitary flows are not known. Just as for the ϕ_{21}/ϕ_{15} flows with $|I| > 1$, the monotonicity property is lost. Typical cases are illustrated in figures 7 and 8.

6 Conclusions

We hope to have demonstrated the rich structure of flows that can be studied by means of a relatively simple nonlinear integral equation. A number of unexpected features have emerged, most notably the consistent failure of the effective central charges to be monotonic functions of the system size. Indeed, looking at the full set of minimal models,

**we understand that this has already been checked by Alyosha Zamolodchikov [51]

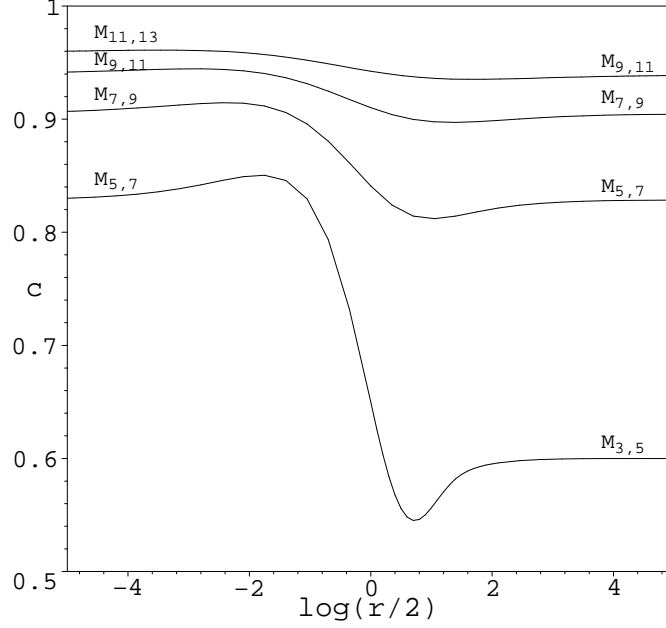


Figure 7: One of the $J = 2$ nonunitary ϕ_{13} perturbed flows calculated using the NLIE of [20].

both unitary and nonunitary, we see that for massless flows a monotonic behaviour of $c_{\text{eff}}(r)$ is very much the exception rather than the rule.

As for future work, the following directions seem natural:

- A more detailed study of both the massive and the massless $a_2^{(2)}$ -related nonlinear integral equations is warranted. This should include a more detailed look at their analytic properties, and further comparisons with perturbed conformal field theories. In addition, the equations should be generalised to describe excited states.
- The type II conjecture remains a curious observed fact, and a deep understanding seems still to be lacking. In cases suffering from this ambiguity, a knowledge of $c_{\text{eff}}(r)$ will never suffice to disentangle the possible destinations of the flows. It would be reassuring to confirm that these models behave as conjectured by some other method.
- The work of [20] was based in part on a massless S-matrix. Here we avoided this aspect, but it would be interesting to find an S-matrix description of the new flows. One might also hope to find integrable lattice models which would yield the massless flows in their continuum limits.
- As discussed in §3, we were unable to treat some flows, due to singularities in the kernel $\phi(\theta)$ crossing the integration contour. It would be worthwhile to study the NLIE in these zones where ξ falls below 1, perhaps by analytic continuation. It would also be interesting to study the NLIE of [20] for $\zeta < 1$, where a similar difficulty is encountered.

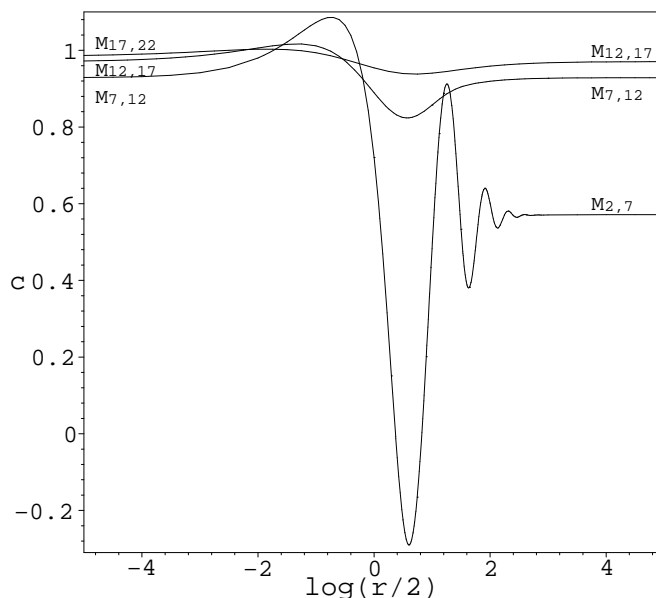


Figure 8: One of the nonunitary ϕ_{13} perturbed flows with index $J = 5$ calculated using the NLIE of [20].

- The method described in §2 should enable a massless nonlinear integral equation to be obtained from any of the (massive) ADE-related systems described in [52, 53]. It would be interesting to know whether these share the same strange properties that we have observed here in the examples related to $a_2^{(2)}$ and $a_1^{(1)}$.
- As mentioned in §4, the initial discovery of the flows (1.1) and (1.2) in [12] came via a staircase model. The new flows also fit naturally into staircase-like patterns, and it is natural to ask whether the existing set of known staircase models [12, 34, 54–56] could be enlarged so as to include these cases.
- Finally, it remains an open question to find TBA equations describing the non-unitary ϕ_{13} flows, or any of the new ϕ_{21} or ϕ_{15} flows. Why is there no TBA for any non-monotonic case? And what has this to do with the monstrosity^{††}, if anything?

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^{††}see [20]

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